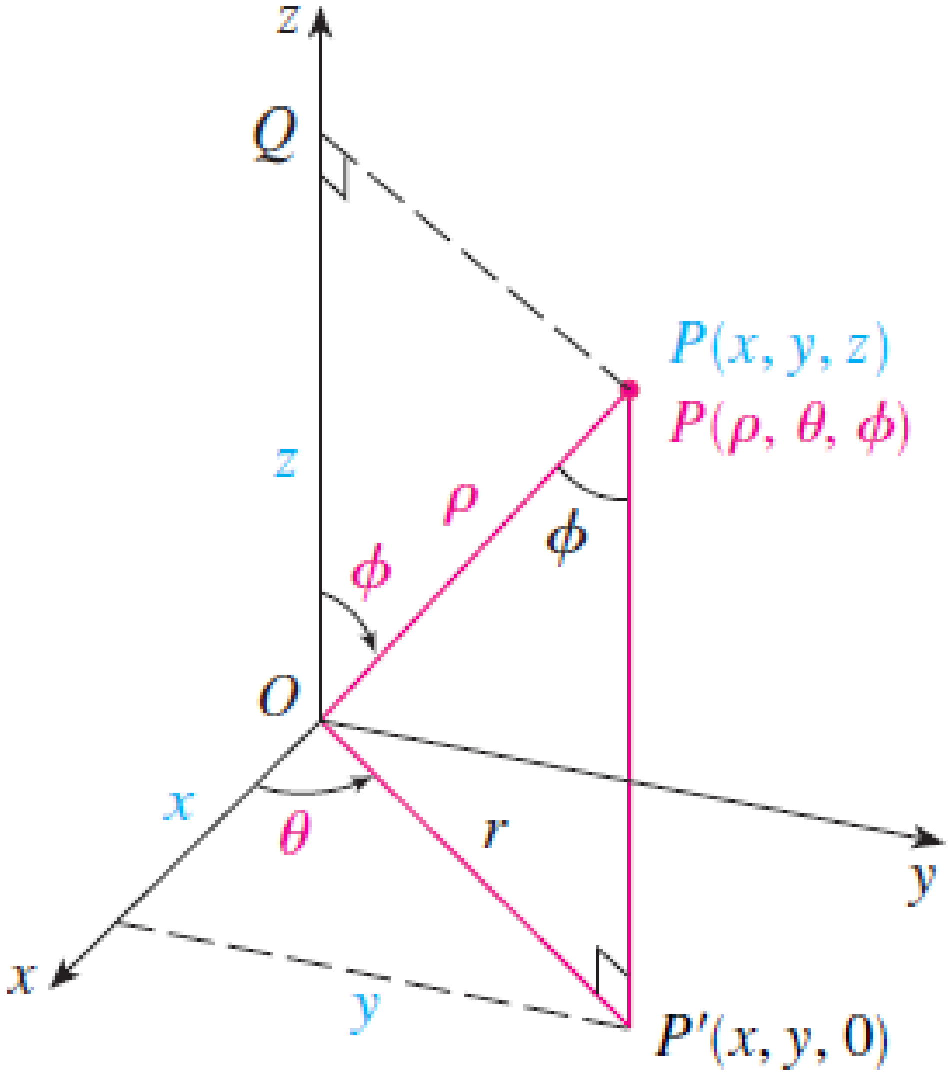
Coordenadas Esféricas



$$x = \rho \operatorname{sen} \phi \cos \theta$$

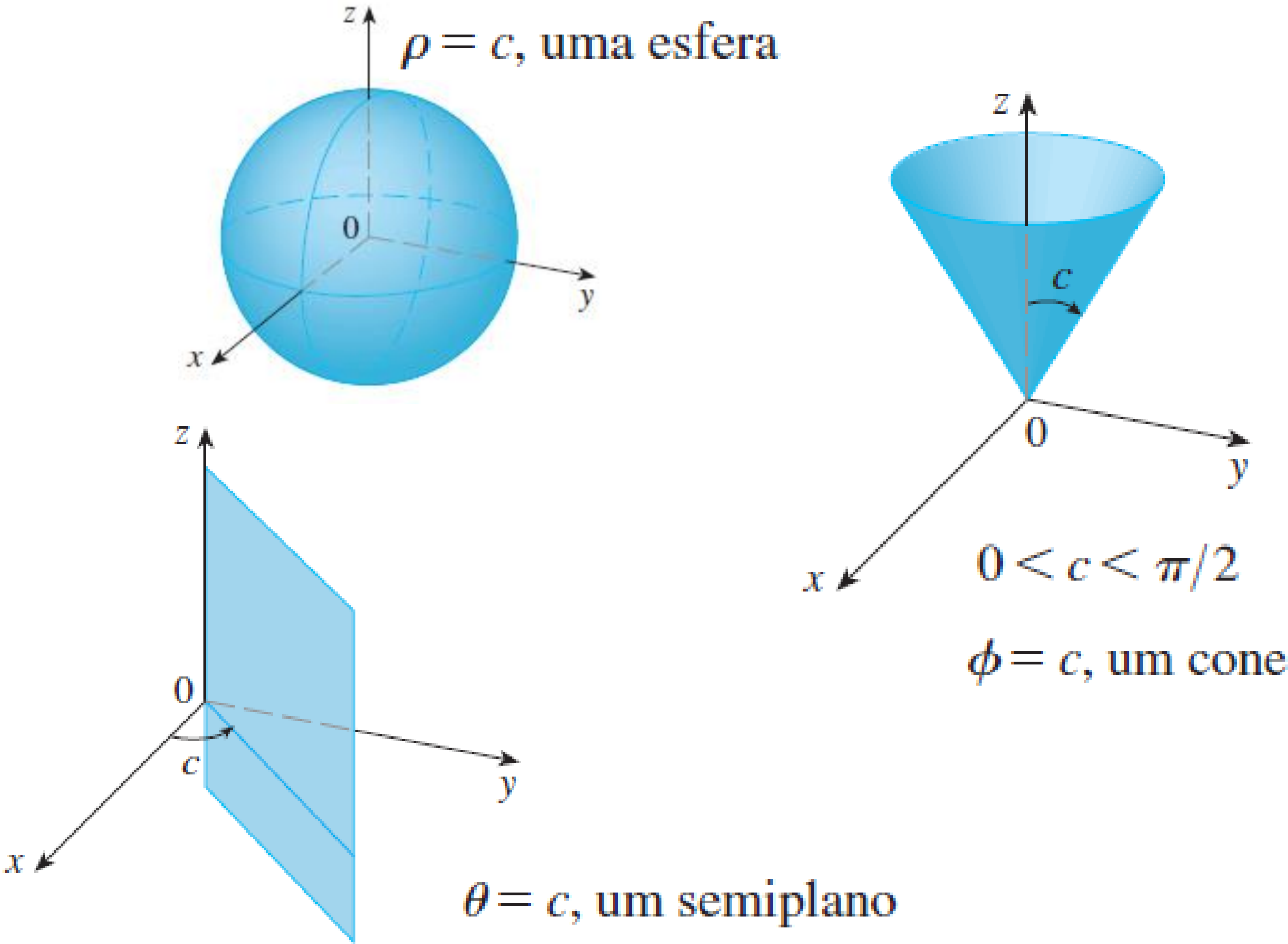
$$y = \rho \operatorname{sen} \phi \operatorname{sen} \theta$$

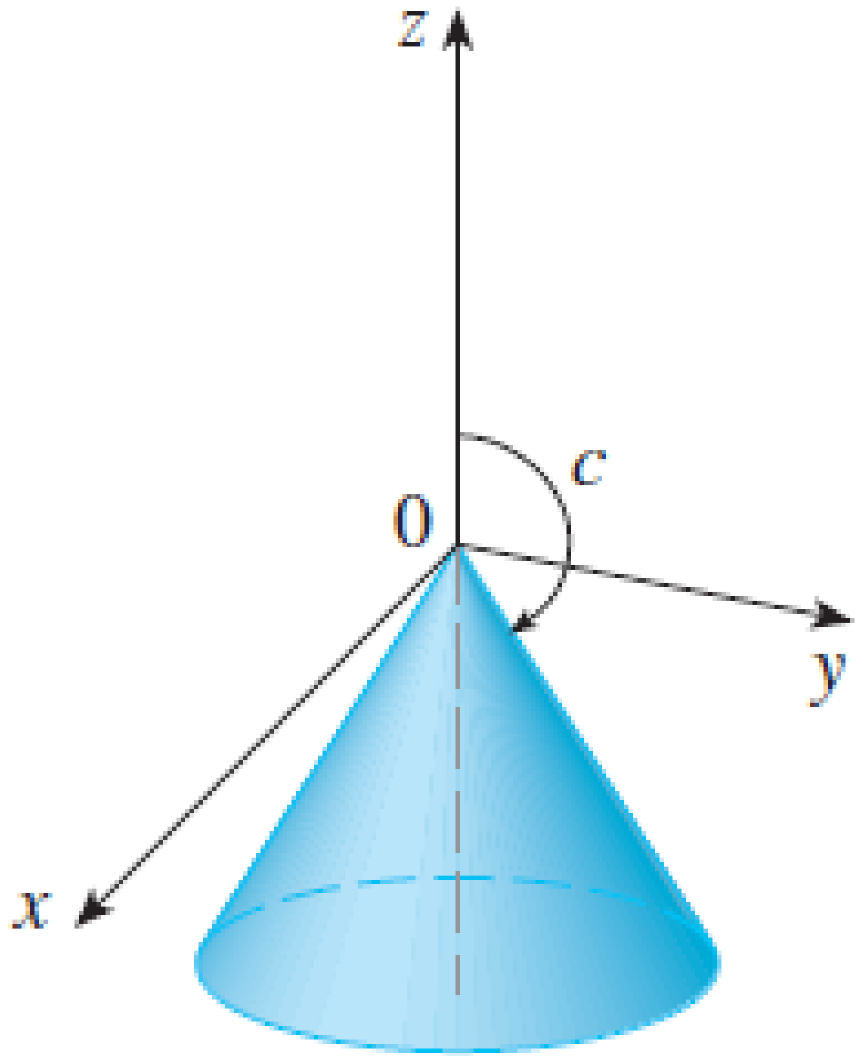
$$z = \rho \cos \phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

As **coordenadas esféricas** (ρ, θ, ϕ) de um ponto P no espaço

$$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$$

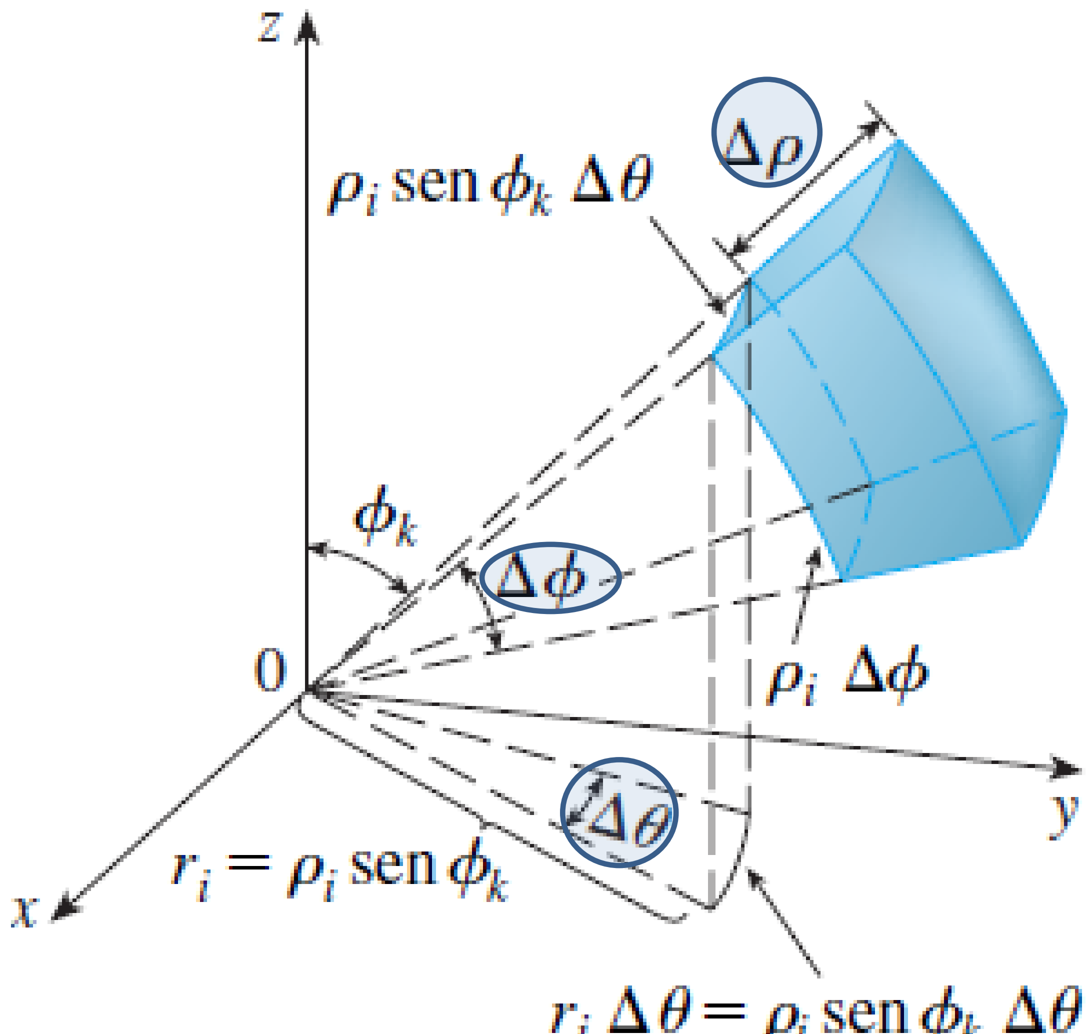

z
ρ = c, uma esfera
0
y
x
z
c
0
y
x
0 < c < π/2
ϕ = c, um cone
z
0
c
y
x
θ = c, um semiplano



$\pi/2 < c < \pi$

Cálculo de Integrais Triplas com Coordenadas Esféricas

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$



$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i \, \Delta\phi)(\rho_i \operatorname{sen} \phi_k \, \Delta\theta) = \rho_i^2 \operatorname{sen} \phi_k \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

$$\iiint_E f(x, y, z)\, dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V_{ijk}$$

$$= \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\tilde{\rho}_i \,\text{sen}\, \tilde{\phi}_k \cos \tilde{\theta}_j,\ \tilde{\rho}_i \,\text{sen}\, \tilde{\phi}_k \,\text{sen}\, \tilde{\theta}_j,\ \tilde{\rho}_i \cos \tilde{\phi}_k)\ \tilde{\rho}_i^2 \,\text{sen}\, \tilde{\phi}_k\, \Delta\rho\, \Delta\theta\, \Delta\phi$$

Mas essa é uma soma de Riemann para a função

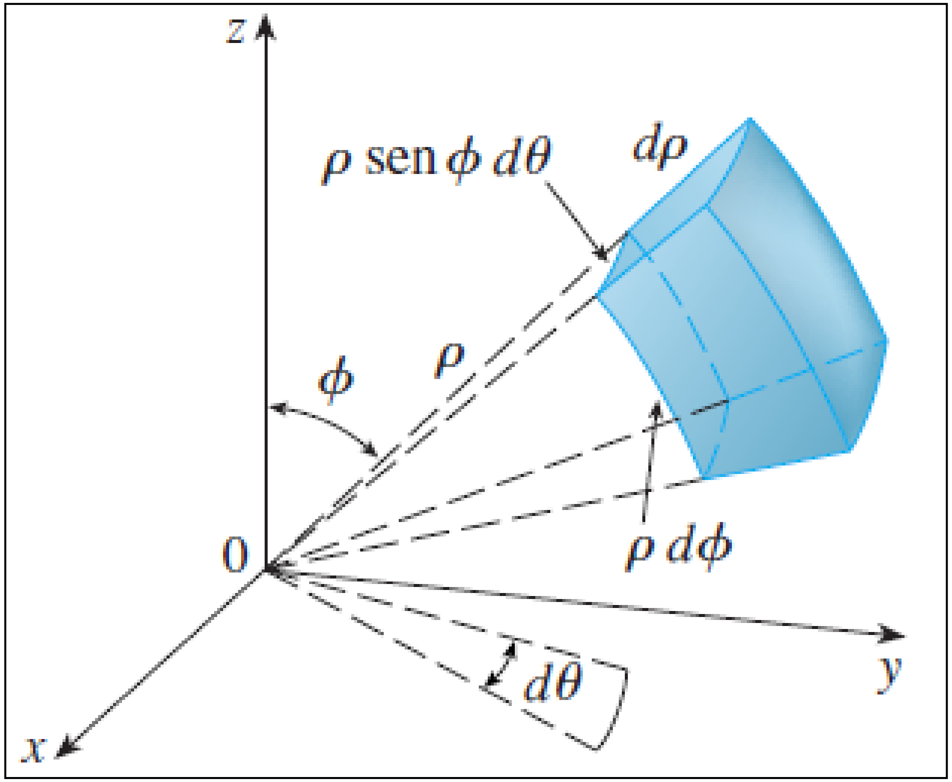
$$F(\rho, \theta, \phi) = f(\rho \,\text{sen}\, \phi \cos \theta,\ \rho \,\text{sen}\, \phi \,\text{sen}\, \theta,\ \rho \cos \phi)\ \rho^2 \,\text{sen}\, \phi$$

3 $\displaystyle\iiint_E f(x, y, z)\, dV$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \operatorname{sen} \phi \cos \theta, \rho \operatorname{sen} \phi \operatorname{sen} \theta, \rho \cos \phi)\, \rho^2 \operatorname{sen} \phi \; d\rho\, d\theta\, d\phi$$

onde E é uma cunha esférica dada por

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$



Calcule $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, onde B é a bola unitária:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

Como o limite de B é uma esfera, utilizaremos coordenadas esféricas:

$$B = \{(\rho, \theta, \phi) \mid 0 \leqslant \rho \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi\}$$

$$x^2 + y^2 + z^2 = \rho^2$$

$$\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} e^{(\rho^2)^{3/2}} \rho^2 \operatorname{sen}\phi \ d\rho\, d\theta\, d\phi$$

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{(\rho^2)^{3/2}}\rho^2 \operatorname{sen}\phi \; d\rho\, d\theta\, d\phi$$

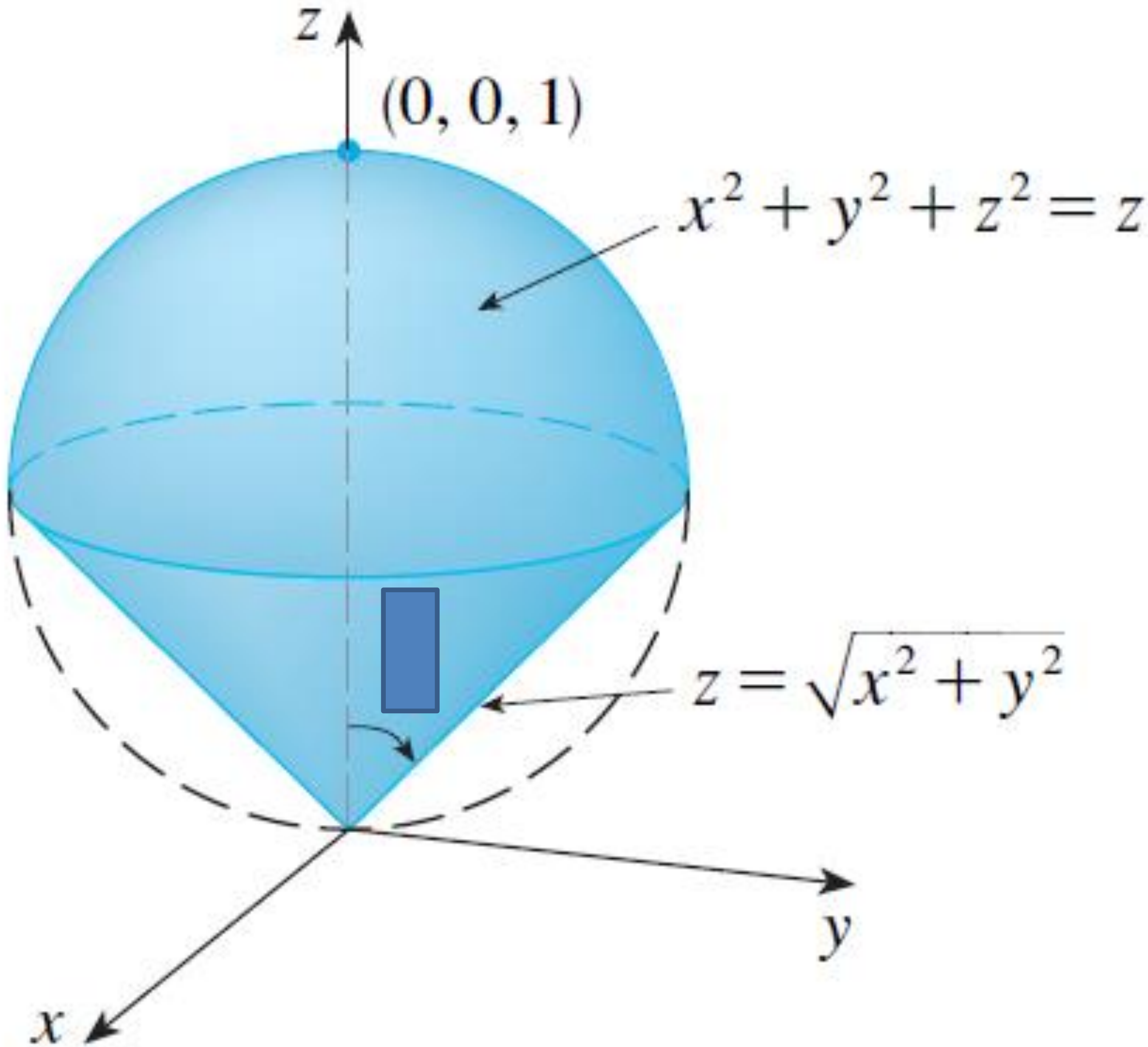
$$= \int_0^{\pi} \operatorname{sen}\phi \; d\phi \int_0^{2\pi} d\theta \int_0^{1} \rho^2 e^{\rho^3}\, d\rho$$

$$= \left[-\cos\phi\right]_0^{\pi}(2\pi)\left[\tfrac{1}{3}e^{\rho^3}\right]_0^{1} = \tfrac{4}{3}\pi(e-1)$$

Com coordenadas retangulares, a integral iterada seria

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$$

volume do sólido



equação da esfera em coordenadas esféricas como

$$\rho^2 = \rho \cos \phi \qquad \text{ou} \qquad \rho = \cos \phi$$

equação do cone pode ser escrita como

$$\rho \cos \phi = \sqrt{\rho^2 \operatorname{sen}^2\phi \, \cos^2\theta + \rho^2 \operatorname{sen}^2\phi \, \operatorname{sen}^2\theta} = \rho \operatorname{sen} \phi$$

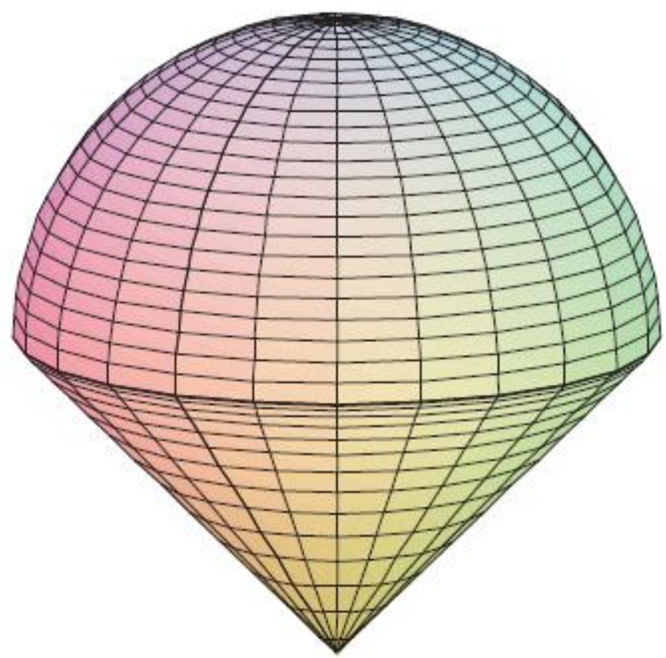
Isto resulta em $\operatorname{sen} \phi = \cos \phi$, ou $\phi = \pi/4$.

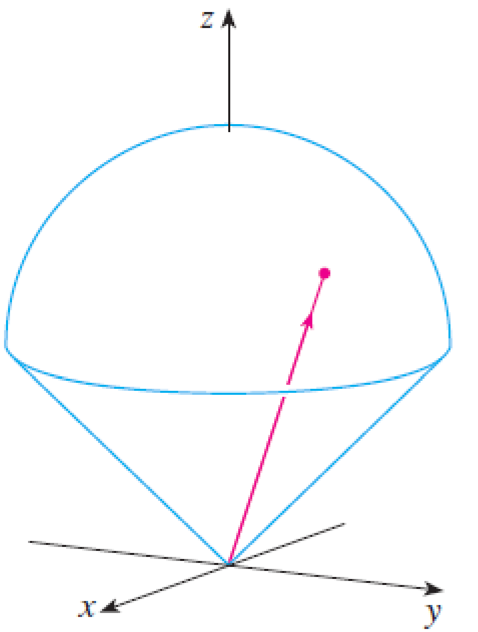
$$E = \{(\rho, \theta, \phi) \mid 0 \leqslant \theta \leqslant 2\pi, \ 0 \leqslant \phi \leqslant \pi/4, \ 0 \leqslant \rho \leqslant \cos \phi\}$$

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \operatorname{sen}\phi \; d\rho \, d\phi \, d\theta$$

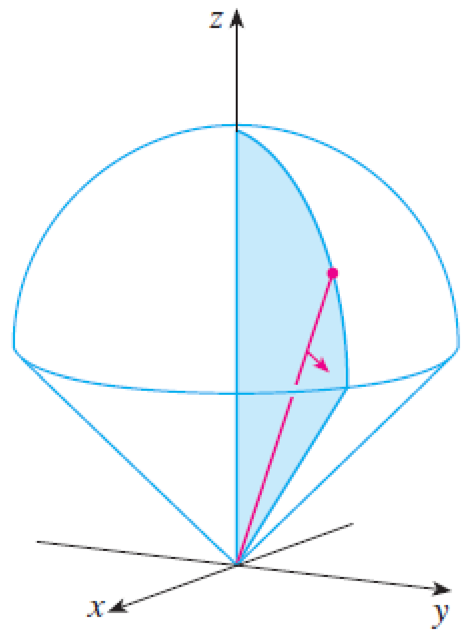
$$= \int_0^{2\pi} d\theta \; \int_0^{\pi/4} \operatorname{sen}\phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \operatorname{sen}\phi \; \cos^3\phi \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8}$$

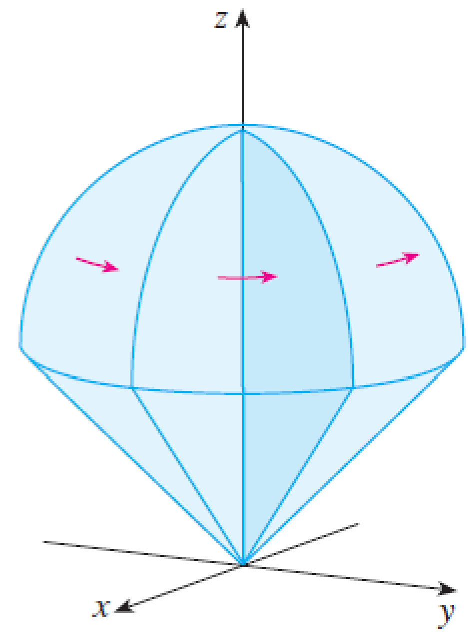




ρ varia de 0 a cos ϕ, enquanto ϕ e θ são constantes.



ϕ varia de 0 a $\pi/4$, enquanto θ é constante.



θ varia de 0 a 2π.