RELATIONS

Let A and B be sets.

A *binary relation* or, simply, *relation* from A to B is a subset of $A \times B$.

(i) $(a, b) \in R$; we then say "a is R-related to b", written aRb.

(ii) $(a, b) \notin R$; we then say "a is not R-related to b", written $a\not{R}b$.

Seja $R \subseteq AXB$ uma relação binária.

$Dom(R) = \{x \in A : (x, y) \in R\}$, domínio da relação

$\mathrm{Im}(R) = \{y \in B : (x, y) \in R\}$, imagem da relação

EXAMPLE

1. A={1,2,3} e B={x,y,z} e R={(1,y),(1,z),(3,z)}.

$1 R y,\ 1 R z,\ 3 R y,\ 1 \not R x,\ 2 \not R x,\ 2 \not R y,\ 2 \not R z,\ 3 \not R x,\ 3 \not R z$

$Dom(R) = \{1,3\}, \text{Im}(R) = \{y, z\}$

2. A familiar relation on the set **Z** of integers is "*m* divides *n*."
$m \mid n$ when m divides n. Thus $6 \mid 30$ but $7 \nmid 25$.

Inverse Relation

Let R be any relation from a set A to a set B

The *inverse of R*, denoted by R^{-1}, is the relation from B to A

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

For example, let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$.

Then the inverse of

$R = \{(1, y), (1, z), (3, y)\}$ is $R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$

$Seja\ R \subseteq AXB$ uma relação binária.

$Dom(R^{-1}) = \{y \in A : (x, y) \in R\},$ domínio da relação

$\text{Im}(R^{-1}) = \{x \in B : (x, y) \in R\},$ imagem da relação

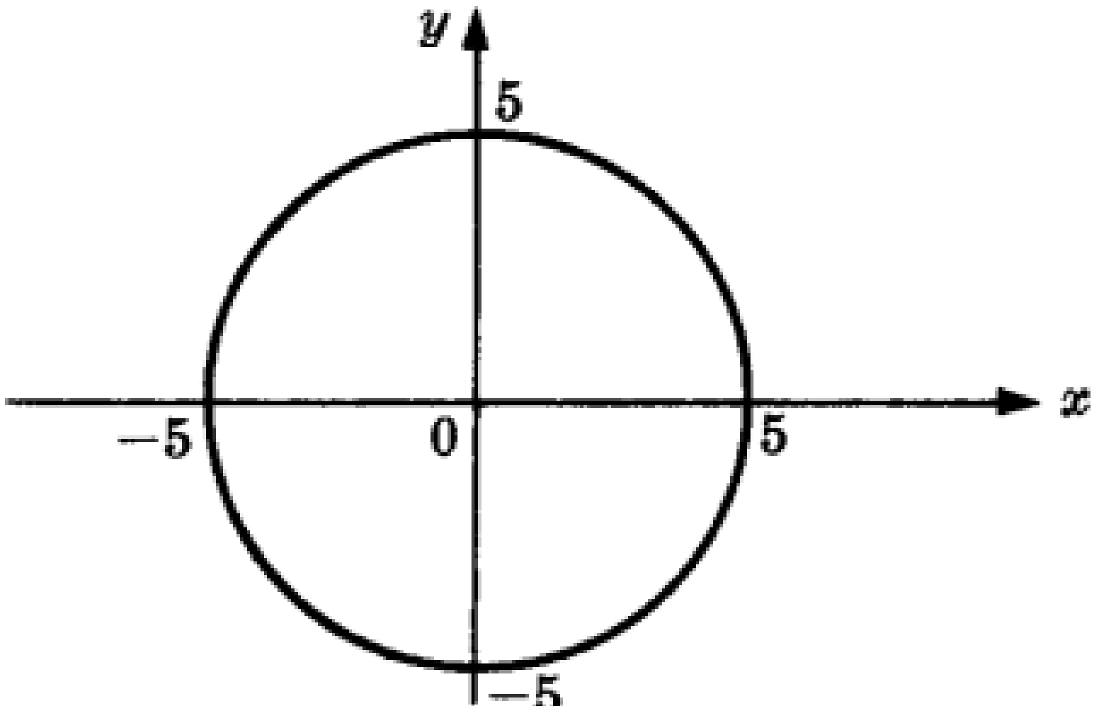
Logo,

$$Dom(R) = \text{Im}(R^{-1})$$

$$\text{Im}(R) = Dom(R^{-1})$$

REPRESENTATIVES OF RELATIONS

Seja $T \subseteq \Re X \Re$ uma relação binária, tal que, "(x,y) $\in$ T see $x^2 + y^2 = 25$".

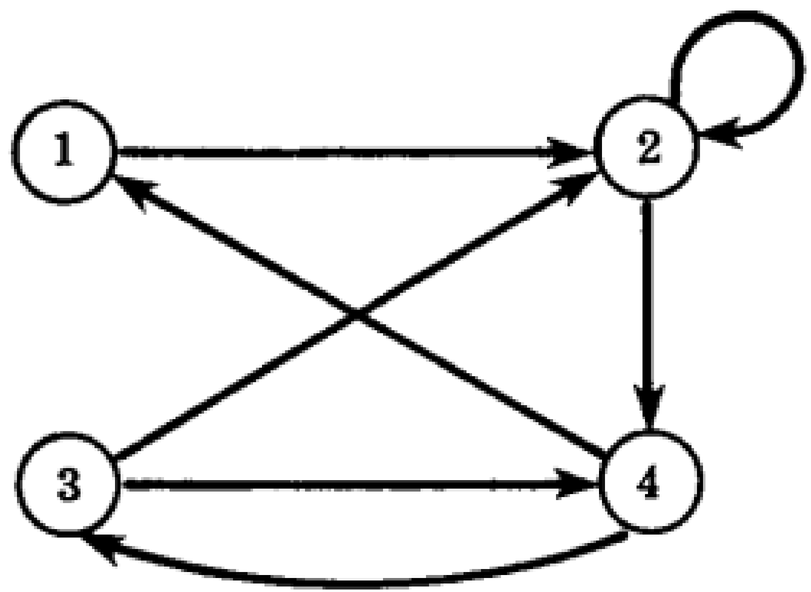


$x^2 + y^2 = 25$

Directed Graphs of Relations on Sets

relation R on the set $A = \{1, 2, 3, 4\}$:

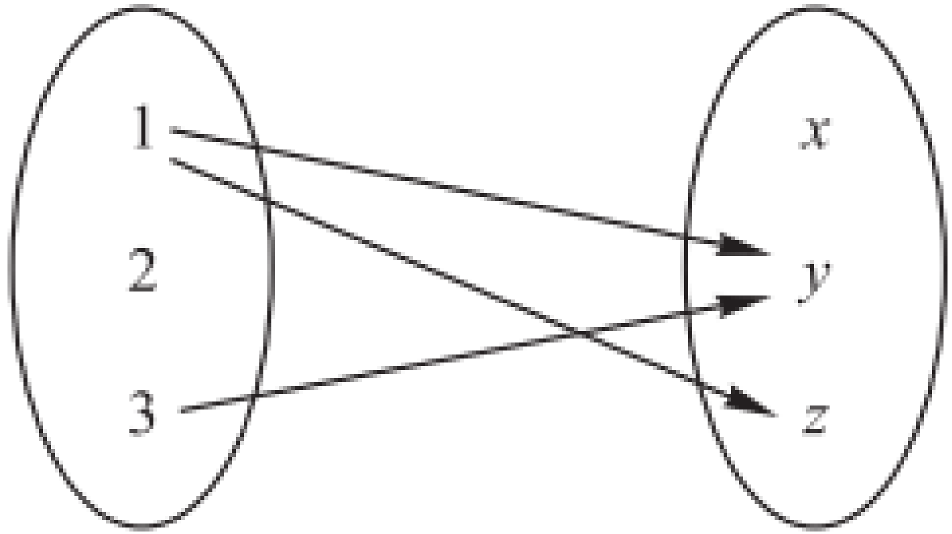
$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$



Pictures of Relations on Finite Sets

$$R = \{(1, y),\ (1, z),\ (3, y)\}$$

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0



0 significa que os elementos não se relacionam

1 significa que os elementos se relacionam

COMPOSITION OF RELATIONS

Let A, B and C be sets,

and let R be a relation from A to B and let S be a relation from B to C.

$$R \subseteq AxB \text{ e } S \subseteq BxC$$

Then R and S give rise to a relation from A to C denoted by $R \circ S$

$$a(R \circ S)c \text{ if for some } b \in B \text{ we have } aRb \text{ and } bSc.$$

$$R \circ S = \{(a, c) \mid \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

$R \circ S$ is called the *composition* of R and S

EXAMPLE

Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let

$R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d, z)\}$

$2(R \circ S)z$ since $2Rd$ and dSz

$3(R \circ S)x$ and $3(R \circ S)z$

$$\boxed{R \circ S = \{(2, z), (3, x), (3, z)\}}$$

Our first theorem tells us that composition of relations is associative.

Theorem

Let A, B, C and D be sets.
Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

Composition of Relations and Matrices

Let M_R and M_S denote respectively the matrix representations of the relations R and S. Then

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$R = \{(1,a),(2,d),(3,a),(3,b),(3,d)\}$$

$$S = \{(b,x),(b,z),(c,y),(d,z)\}$$

$$RoS = \{(2,z),(3,x),(3,z)\}$$

Multiplying M_R and M_S we obtain the matrix

$$M = M_R M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$RoS = \{(2,z),(3,x),(3,z)\}$$

Entradas não nulas na matriz significam que os elementos se relacionam.

TYPES OF RELATIONS

A relation R on a set A is *reflexive* if aRa for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. Thus R is not reflexive if there exists $a \in A$ such that $(a, a) \notin R$.

A relation R on a set A is *symmetric* if whenever aRb then bRa, that is, if whenever $(a, b) \in R$ then $(b, a) \in R$.
Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

A relation R on a set A is *antisymmetric* if whenever aRb and bRa then $a = b$, that is, if $a \neq b$ and aRb then $b \not R a$.
Thus R is not antisymmetric if there exist distinct elements a and b in A such that aRb and bRa.

A relation R on a set A is *transitive* if whenever aRb and bRc then aRc, that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$.

Thus R is not transitive if there exist $a, b, c \in R$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

EXAMPLE

Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$
$R_2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
$R_3 = \{(1, 3), (2, 1)\}$
$R_4 = \emptyset$, the empty relation
$R_5 = A \times A$, the universal relation

Determine which of the relations are reflexive.
Determine which of the relations are symmetric.
Determine which of the relations are antisymmetric.
Determine which of the relations are transitive.

EXAMPLE Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set $\mathbf{Z}$ of integers.

(2) Set inclusion $\subseteq$ on a collection C of sets.

(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.

(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.

(5) Relation $|$ of divisibility on the set $\mathbf{N}$ of positive integers.
(Recall $x \mid y$ if there exists z such that $xz = y$.

Determine a validade ou não das propriedades reflexixa, simétrica, antissimétrica e transitiva das relações acima.

EQUIVALENCE RELATIONS

Consider a nonempty set S.
A relation R on S is an *equivalence relation* if R is reflexive, symmetric, and transitive.

That is, R is an equivalence relation on S if it has the following three properties:

(1) For every $a \in S$, aRa.

(2) If aRb, then bRa.

(3) If aRb and bRc, then aRc.

EXAMPLE

(a) Let L be the set of lines and let T be the set of triangles in the Euclidean plane.

(i) The relation "is parallel to or identical to" is an equivalence relation on L.

(ii) The relations of congruence and similarity are equivalence relations on T.

(b) The relation $\subseteq$ of set inclusion is not an equivalence relation.
It is reflexive and transitive, but it is not
symmetric since $A \subseteq B$ does not imply $B \subseteq A$.

Propriedades de Relações

Seja R uma relação definida em um conjunto A.

- Se, para todo $x \in A$, temos $x \, R \, x$, dizemos que R é *reflexiva*.
- Se, para todo $x \in A$, temos $x \not R \, x$, dizemos que R é *anti-reflexiva*.
- Se, para todo $x, y \in A$, temos $x \, R \, y \Rightarrow y \, R \, x$, dizemos que R é *simétrica*.
- Se, para todo $x, y \in A$, temos $(x \, R \, y \wedge y \, R \, x) \Rightarrow x = y$, dizemos que R é *anti-simétrica*.
- Se, para todo $x, y, z \in A$, temos $(x \, R \, y \wedge y \, R \, z) \Rightarrow x \, R \, z$, dizemos que R é *transitiva*.